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ON THE RATE OF CONVERGENCE IN STRASSEN'S FUNCTIONAL LAW
OF THE ITERATED LOGARITHM

by

Joop Mijnheer

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180. R. Brigola, Stochastic filtering solutions for ill-posed linear problems and their extension to measurable transformations, Mar. 87.

181. G. Samorodnitsky, Maxima of symmetric stable processes, Mar. 87.

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183. H.L. Hurd, Nonparametric time series analysis for periodically correlated processes, Apr. 87. *IEEE Trans. Information Theory*, to appear.

184. T. Mori and H. Oodaira, Freidlin-Wentzell estimates and the law of the iterated logarithm for a class of stochastic processes related to symmetric statistics, May 87. *Yokohama Math. J.*, to appear.

185. R.F. Serfozo, Point processes, May 87. *Operations Research Handbook on Stochastic Processes*, to appear.

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198. T. Hsing, On the extreme order statistics for a stationary sequence, July 87. *Stochastic Proc. Appl.* 29, 1988, 155-169.

199. T. Hsing, Characterization of certain point processes, Aug. 87. *Stochastic Proc. Appl.* 26, 1987, 297-316.

200. J.P. Nolan, Continuity of symmetric stable processes, Aug. 87. *J. Multivariate Anal.*, 1988, to appear.

201. M. Marques and S. Cambanis, Admissible and singular translates of stable processes, Aug. 88.

202. O. Kallenberg, One-dimensional uniqueness and convergence criteria for exchangeable processes, Aug. 87. *Stochastic Proc. Appl.* 28, 1988, 159-183.

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218. R.F. Serfozo, Poisson functionals of Markov processes and queueing networks, Dec. 87.

219. J. Bather, Stopping rules and ordered families of distributions, Dec. 87. *Sequential Anal.*, 7, 1988, 111-126.

220. S. Cambanis and M. Maejima, Two classes of self-similar stable processes with stationary increments, Jan. 88.

221. H.P. Hucke, G. Kallianpur and R.L. Karandikar, Smoothness properties of the conditional expectation in finitely additive white noise filtering, Jan. 88. *J. Multivariate Anal.*, to appear.

ON THE RATE OF CONVERGENCE IN STRASSEN'S FUNCTIONAL LAW OF THE ITERATED LOGARITHM

by

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Summary

An improvement of the rate of convergence in the functional law of the iterated logarithm (F.L.I.L) is given.

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Key words and phrases: rate of convergence, functional limit theorem.

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1. Introduction.

Let $\{W(t)\colon 0 \le t < \infty\}$ be a Brownian motion or a Wiener process on $(\Omega,\mathcal{F},P)$. We define the functions $\{f_T\colon T \ge 20\}$

$$f_T\colon [0,1] \times \Omega \to \mathbb{R}$$

by

$$f_T(t,\omega) = W(Tt,\omega)\ (2T \log \log T)^{-1/2}.$$

Let K be the subset of absolutely continuous functions $x \in C[0,1]$ such that

$$x(0) = 0$$

and

$$\int_0^1 \{\dot{x}(t)\}^2 dt \le 1.$$

The functional law of the iterated logarithm (F.L.I.L.) states

<u>Theorem</u>. (Strassen 1964) W.p.1. $\{f_n\colon n \in \mathbb{N}\}$ is relatively compact with limit set K.

Two results about the rate of convergence are known.

$$P(d_\infty(f_n,K) \ge (\log \log n)^{-\alpha} \text{ i.o.}) = \begin{cases} 1 & \text{according as } \alpha \ge 1, \\ 0 & \phantom{\text{according as }} \alpha < 1/2, \end{cases}$$

(Bolthausen 1978) and the following sharpening of the foregoing result.

$$P(d_\infty(f_T,K) \ge (\log \log T)^{-\alpha} \text{ i.o.}) = \begin{cases} 1 & \text{according as } \alpha > 2/3, \\ 0 & \phantom{\text{according as }} \alpha < 2/3, \end{cases}$$

(Grill 1987). The distance d_∞ is the usual distance in $C[0,1]$ and will be

defined in section 2.

The main result of this paper will be an improvement of this last result.

Theorem. Let $\{f_n\}$ and K be defined as above. Then

$$P(d_\infty(f_n,K) \geq \psi_\delta(n) \quad \text{i.o.}) = \begin{cases} 1 & \\ & \text{according as} \\ 0 & \end{cases} \quad \begin{matrix} \delta < 0, \\ \\ \delta > 0, \end{matrix}$$

where

$$\psi_\delta(n) = \frac{1}{6}(1+\delta)(\log_3 n)(\log_2 n)^{-2/3}.$$

A simple proof of Strassen's F.L.I.L. is given in Chover (1967). A detailed proof can be found in the book of Freedman (1971). An extensive discussion of the different formulations of functional laws of the iterated logarithm is given in Taqqu and Czado (1985). In section 5 we compare the approach in this paper with several other approaches.

2. Some results of Gaussian processes.

The set K in Strassen's F.L.I.L. is the unit ball of the Hilbert space

$$H = \{f: [0,1] \to \mathbb{R}, \quad f(t) = \int_0^t \dot{f}(s)ds, \quad \int_0^1 \{\dot{f}(s)\}^2 ds < \infty\}$$

with inner product

$$(f,g) = \int_0^1 \dot{f}(s)\dot{g}(s)ds, \qquad f,g \in H.$$

The sequence $\{\varphi_n: n=0,1,\ldots\}$, where

$$\varphi_n(t) = \frac{2\sqrt{2}}{\pi(2n+1)} \sin((2n+1)\pi t/2) \qquad 0 \leq t \leq 1,$$

is a complete orthonormal system in H. Let $\{X_n: n=0,1,\ldots\}$ be a sequence of i.i.d. N(0,1) distributed random variables. The Karhunen-Loève expansion of the Brownian motion $\{W(t): 0 \leq t \leq 1\}$ states

$$W(t) = \sum_{n=0}^{\infty} X_n \varphi_n(t). \tag{2.1}$$

For more details see Loève (1963) or Jain and Marcus (1978). In theorem 3.3 of the last paper they prove that the expansion (2.1) converges uniformly a.s. The space H is also used by Kuelbs and LePage (1973) in order to prove functional laws.

In our theorem we use two norms. The sup norm in C[0,1]

$$\|f\|_\infty = \sup_{0\le t\le 1} |f(t)|$$

and the norm in H

$$\|f\|_H = \{ \int_0^1 \{\dot{f}(s)\}^2 ds\}^{1/2}.$$

We have the following relation between these two norms

$$\|f\|_\infty \le \|f\|_H.$$

The corresponding metrics are denoted by d_∞ and d_H. For each natural number m we define the (Gaussian) processes $\{U_m(t)\colon 0\le t\le 1\}$ and $\{V_m(t)\colon 0\le t\le 1\}$ by

$$V_m(t) = \sum_{n=0}^{m-1} X_n \varphi_n(t)$$

and

$$U_m(t) + V_m(t) = W(t),$$

i.e.

$$U_m(t) = \sum_{n=m}^{\infty} X_n \varphi_n(t).$$

Then we have

$$EU_m(t) = EV_m(t) = 0,$$

$$\sigma^2(U_m(t)) = \sum_{n=m}^{\infty} \varphi_n^2(t) \quad \text{and} \quad \sigma^2(V_m(t)) = \sum_{n=0}^{m-1} \varphi_n^2(t).$$

In the case t=1 we have for $m \to \infty$

(2.2a) $$\sigma^2(V_m(1)) = 2\pi^{-2} \sum_{n=0}^{m-1} 4(2n+1)^{-2} = 1 - \frac{2}{\pi^2 m} + O(m^{-2})$$

and

(2.2b) $$\sigma^2(U_m(1)) = \frac{2}{\pi^2 m} + O(m^{-2}).$$

We also have

$$\| V_m \|_H^2 = 2 \sum_{n=0}^{m-1} X_n^2 \int_0^1 \cos^2\{(2n+1)\pi t/2\}dt \qquad = \sum_{n=0}^{m-1} X_n^2 .$$

Thus $\| V_m \|_H^2$ has a chi-square distribution with m degrees of freedom.

We note that U_m and V_m are independent Gaussian processes. There exists a rich literature on the maximum of Gaussian processes. See for example Berman (1985), Talagrand (1988), Piterbarg and Prisjaznjuk (1978) and the references in those papers. We shall not use the results out of one of these papers but prove the following lemma. Because the processes U_m and V_m have such a nice structure we give new proofs. (Of course making use of the ideas of the other papers.) Note that the processes have no independent increments. It is easy to see that the variance takes its maximum value for t=1. The processes will reach their maximum near t=1, as we can conclude from the following lemma. In this lemma we shall compare the tail of distribution of the maximum with the tail of the distribution of the process at t=1.

Lemma 2.1. Let the Gaussian processes U_m and V_m be defined as above. Let $\epsilon > 0$. Then we have for $m,u \to \infty$ and $m = o(u)$

(2.3.a) $$P(\max_{0\le t\le 1} V_m(t) > u) \le (1 + o(1))P(V_m(1) > u(1-\epsilon))$$

and

(2.3.b) $$P(\max_{0\le t\le 1} U_m(t) > u) \le (1 + o(1))\, P(U_m(1) > u(1-\epsilon)).$$

The distribution of the maximum of a Brownian motion is given by

$$(2.4) \qquad P(\sup_{0\le t\le 1} W(t) > u) = 2P(W(1) > u)$$

See Freedman (1971) corollary 29. Thus, for $u \to \infty$, we have

$$P(\sup_{0\le t\le 1} W(t) > u) \sim P(W(1) > u - u^{-1}\log 2)$$

by application of the expansion

$$(2.5) \qquad P(W(1) > u) \sim (2\pi)^{-1/2} u^{-1} e^{-u^2/2} \qquad \text{for } u \to \infty.$$

See Freedman (1971) lemma (4.a).

In (2.3.a) we have the trivial lower bound

$$(2.6) \qquad P(V_m(1) > u) \le P(\max_{0\le t\le 1} V_m(t) > u) \qquad \text{for all } u.$$

Similarly for U_m.

<u>Proof of lemma 2.1</u>.

<u>Part a</u>. The mean value theorem implies

$$V_m(t+h) = V_m(t) + \sqrt{2}\, h \sum_{k=0}^{m-1} X_k \cos\{(2k+1)\tfrac{\pi}{2}(t+h_t)\}$$

when mh is small. Divide the interval [0,1] in Δ^{-1} (integer) intervals of length Δ. Then we have

$$(2.7) \qquad \begin{aligned} P(\max_{0\le t\le 1} V_m(t) > u) &\le \sum_{j=1}^{\Delta^{-1}-1} P(\max_{j\Delta\le t\le (j+1)\Delta} V_m(t) > u) \\ &= \sum_{j=0}^{\Delta^{-1}-1} P(V_m(j\Delta) + \max_{j\Delta\le t\le (j+1)\Delta} (V_m(t) - V_m(j\Delta)) > u). \end{aligned}$$

We also have

$$\max_{j\Delta \le t \le (j+1)\Delta} | V_m(t)-V_m(j\Delta)| \le \Delta \sqrt{2} \sum_{k=0}^{m-1} |X_k|.$$

Note that this (random) bound is independent of t and j. For $\Delta^2 m$ small we can apply the central limit theorem in order to obtain

$$(2.8) \qquad P(\max_{j\Delta \le t \le (j+1)\Delta} V_m(t) > u) \le P(V_m(j\Delta) > u-\epsilon u) + r_j$$

where the error r_j is asymptotically small with respect to $P(V_m(1) > u-\epsilon u)$ for $u \to \infty$. Using (2.7), $\sigma^2(V_m(t)$ is maximal for t=1, (2.8) and (2.5) we obtain

$$\begin{aligned} P(\max_{0\le t\le 1} V_m(t) > u) &\le \Delta^{-1}P(V_m(1) > u-\epsilon u) + \Sigma r_j \\ &\le P(V_m(1) > u(1-2\epsilon)). \end{aligned}$$

<u>Part b</u>. $U_m(t)$ is an infinite series. We write

$$U_m(t) = \sum_{k=m}^{m^2-1} \frac{2\sqrt{2}}{\pi(2k+1)} X_k \sin(2k+1) \frac{\pi}{2} t + U_{m^2}(t).$$

We have seen that $\sigma^2(U_{m^2}(t)) \le cm^{-2}$ uniformly in t. Thus we obtain (uniformly in t)

$$P(|U_{m^2}(t)| > u\delta) = o(P(U_m(1) > u(1-\epsilon))$$

for $m,u \to \infty$.

The further proof is similar to that of part a. □

We apply the following asymptotic expansion for the right tail of the chi-square distribution with increasing degrees of freedom.

<u>Lemma 2.2</u>. For $m \to \infty$ and $m = o(x)$ for $x \to \infty$ we have

$$P(\chi_m^2 > x) = \{1 + o(1)\}\ \{\pi\ m\}^{-1/2}\ e^{-1/2 x}\ e^{(-1/2 m-1)\log(xe/m)}.$$

Proof of Lemma 2.2.

The assertion follows easily after some calculus and the application of Stirling's formula. □

The following assertions are well-known for gamma distributions but we shall only apply them for chi-square distributions.

Lemma 2.3.

Let X (resp. Y) be χ_n^2 (resp. χ_m^2) distributed. X and Y are independent. Then

a) X + Y has a χ_{n+m}^2 distribution

b) X + Y and X/(X + Y) are independent

c) X/(X + Y) has a $B(\frac{n}{2}, \frac{m}{2})$ distribution.

Proof.

See Rohatgi (1976) Section 5.3 Theorem 4 resp. Th. 6 and Th. 15. □

Define the projection Π_m of H onto the finite-dimensional subspace with base $\{\varphi_0, \ldots, \varphi_{m-1}\}$ by

$$\Pi_m\left(\sum_{n=0}^{\infty} a_n\varphi_n(t)\right) = \sum_{n=0}^{m-1} a_n\varphi_n(t).$$

Thus we have

$$V_m = \Pi_m W.$$

We use the notation Lx resp. L_2x for log x resp. log log x.

3. Lower bound.

Define the sequence $n_k = \exp(k\varphi(k))$ where φ is slowly varying at infinity and $\lim_{n\to\infty} \varphi(x) = \infty$. Then we have, for $k \to \infty$, $n_k/n_{k+1} \sim \exp(-\varphi(k)) \to 0$ and

$$P((2n_{k+1}\, L_2 n_{k+1})^{-\frac{1}{2}} \| W(n_k \cdot) \|_\infty > \epsilon\, \psi_0(n_k)) =$$

$$\leq 4P(|U| > \epsilon\psi_0(n_k)(2\, L_2 n_{k+1}\, (n_{k+1}/n_k))^{\frac{1}{2}})$$

using (2.4) and the scaling property of a Brownian motion

$$= O(e^{-\epsilon^2 \psi_o^2(n_k) e^{\varphi(k)} L_2 n_k}\, (\psi_o(n_k))^{-1} (L_2 n_k)^{-\frac{1}{2}} e^{-\frac{1}{2}\varphi(k)})$$

by (2.5). We choose $\varphi(k)$ such that the summation of these probabilities converges. For example take $\varphi(k) = 4/3\, L_2 k$.

Now we define for $k=1,2,\ldots$. the sequence of functions f_k^*

$$f_k^* : [0,1] \times \Omega \to \mathbb{R}$$

by

$$f_k^*(t,\omega) = (2n_{k+1} L_2 n_{k+1})^{-\frac{1}{2}} \{W((n_{k+1} - n_k)t + n_k, \omega) - W(n_k, \omega)\}.$$

We easily see that $f_k^*(t)$ has the same distribution as $(2n_{k+1} L_2 n_{n+1})^{-\frac{1}{2}}$ $(1-n_k/n_{k+1})^{\frac{1}{2}} W(n_{k+1} t)$. We can write, for each k,

$$f_k^*(t) = \sum_{j=0}^{\infty} X_{k,j}^* \, \varphi_j(t) \cdot \{2L_2 n_{k+1}\}^{-\frac{1}{2}} \{1 - n_k/n_{k+1}\}^{\frac{1}{2}}$$

where $X_{k,j}^*$, $j=0,1,\ldots,$ are i.i.d. $N(0,1)$ distributed random variables. Remark that the random variables $X_{k,j}^*$ depend on k. Take $m_k = (L_2 n_k)^{1/3}$. Then we have

$$P(\| \Pi_{m_k} f_k^* \|_H > 1 + \psi_\delta(n_k)) \tag{3.1}$$

$$= P(\chi_{m_k}^2 > (2L_2 n_{k+1})\, (1+\psi_\delta(n_k))^2\, (1-n_k/n_{k+1})^{-1})$$

which can be estimated using lemma 2.2. For $\delta < 0$ we have

$$\sum_k P(\| \Pi_{m_k} f_k^* \|_H > 1 + \psi_\delta(n_k)) = \infty.$$

Using the property that the increments of a Wiener process are independent, the Borel-Cantelli lemma implies that

$$P(\| \Pi_{m_k} f_k^* \|_H > 1 + \psi_\delta(n_k) \quad \text{i.o.}) = 1.$$

Therefore, w.p. 1 we have

$$\Pi_{m_k} f_k^* \notin K \qquad \text{i.o.}$$

For $\Pi_{m_k} f_k^* \notin K$ the projection onto K is given by $\| \Pi_{m_k} f_k^* \|_H^{-1} \; \Pi_{m_k} f_k^*$. Thus w.p. 1 we have

$$d_H(\Pi_{m_k} f_k^*, K) = d_H(\Pi_{m_k} f_k^*, \| \Pi_{m_k} f_k^* \|_H^{-1} \Pi_{m_k} f_k^*) = \| \Pi_{m_k} f_k^* \|_H - 1 > \psi_\delta(n_k) \quad \text{i.o.}$$

Now we shall show that w.p. 1 we have, for $0 < \delta_1 < \delta$, $d_\infty(f_k^*, K) > (1-\delta_1)\psi_o(n_k)$ i.o. Similarly as above we can show that for $\delta > 0$

$$\sum_k P(\| \Pi_{m_k} f_k^* \|_H > 1 + (1 + \delta)\psi_o(n_k)) < \infty.$$

Then the Borel-Cantelli lemma implies

$$P(\| \Pi_{m_k} f_k^* \|_H > 1 + (1+\delta)\psi_o(n_k) \quad \text{i.o.}) = 0.$$

Define the events A_k, $k = 1,2,\ldots$ by

$$1 + (1-\delta)\psi_o(n_k) \le \| \Pi_{m_k} f_k^* \|_H \le 1 + (1+\delta)\psi_o(n_k) \tag{3.2}$$

or

$$(2L_2 n_k)(1 + (1-\delta)\psi_o(n_k))^2 (1 - n_k n_{k+1}^{-1})^{-2} \le \sum_{j=0}^{m_k - 1} (X_{k,j}^*)^2 \le (2L_2 n_k) \cdot$$

$$\cdot\ (1 + (1+\delta)\psi_o(n_k))^2\ (1 - n_k n_{k+1}^{-1})^{-2}\ .$$

Using lemma 2.1. part a we have that, for estimating the right tail probabilities of $\| \Pi_{m_k} f_k^* \|_\infty$, we may use the r.v.

$$\Pi_{m_k} f_k^*(1) = \{ \sum_{j=0}^{m_k-1} X_{k,j}^*\ \varphi_j(1)\}\ \{2L_2 n_{k+1}\}^{-1/2}(1 - n_k n_{k+1}^{-1})^{1/2}.$$

The Cauchy-Schwarz inequality implies

$$(\ \Pi_{m_k} f_k^*(1))^2 \le (1-n_k n_{k+1}^{-1})\{ \sum_{j=0}^{m_k-1} (X_{k,j}^*)^2\}\ \{2L_2 n_{k+1}\}^{-1}\ \{ \sum_{j=0}^{m_k-1} \varphi_j^2(1)\}$$

$$\le \| \Pi_{m_k} f_k^* \|_H^2 \cdot \sigma^2(V_m(1)),$$

where $\sigma^2(V_m(1))$ is given in (2.2.a). The r.v. $\Pi_{m_k} f_k^*(1)$ has a normal distribution with E $\Pi_{m_k} f_k^*(1) = 0$ and

$$\sigma^2(\Pi_{m_k} f_k^*(1)) = \sigma^2(V_m(1))\ (2L_2 n_{k+1})^{-1}(1 - n_k n_{k+1}^{-1}).$$

The vector $X^* = (X_{k,0}^*, \dots, X_{k,\ m_k-1}^*)$ has a m_k-dimensional normal distribution. There exists an orthogonal transformation P such that the first row vector of PX^* becomes $\{\sigma(V_m(1))\}^{-1}\ \{ \sum_{j=0}^{m_k-1} X_{k,j}^*\varphi_j(1)\}$. It follows from lemma 2.3. part c that

$$\{\sigma^2(V_m(1))\}^{-1}\ \{1-n_k n_{k+1}^{-1}\}^{-1}\ \{\Pi_{m_k} f^*(1)\}^2\ \| \Pi_{m_k} f_k^* \|_H^{-2} \tag{3.3}$$

has a B(½, ½m_k - ½) distribution and by lemma 2.3, part b is the r.v. given in (3.3) independent of $\| \Pi_{m_k} f_k^* \|_H^2$.

Consider

$$P(\| \Pi_{m_k} f_k^* \|_\infty > 1 - \epsilon \wedge A_k) = P(\| \Pi_{m_k} f_k^* \|_\infty > 1-\epsilon | A_k)\ P(A_k) =$$

$$\geq P(B(\tfrac{1}{2}, \tfrac{1}{2}m_k - \tfrac{1}{2}) > 1 - \epsilon_1) P(A_k)$$

for some positive ϵ_1. With the estimate

$$P(B(\tfrac{1}{2}, \tfrac{1}{2}m_k - \tfrac{1}{2}) > 1-\epsilon_1) \sim c m_k^{-\frac{1}{2}} e^{-\frac{1}{2}m_k \log \epsilon_1^{-1}}$$

for $k \to \infty$ we obtain

$$\sum_k P(\|\Pi_{m_k} f_k^* \|_\infty > 1 - \epsilon \wedge A_k) = \infty.$$

The Borel-Cantelli lemma implies that w.p.1 we have i.o.

$$\| \Pi_{m_k} f_k^* \|_\infty > 1 - \epsilon \wedge \| \Pi_{m_k} f_k^* \|_H \geq 1 + (1-\delta)\psi_o(n_k).$$

Or, w.p. 1 we have i.o.

$$\| \Pi_{m_k} f_k^* \|_\infty > 1 - \epsilon \wedge d_H(\Pi_{m_k} f_k^*, K) = \| \Pi_{m_k} f_k^* \|_H - 1 > (1-\delta)\psi_o(n_k).$$

Next we want to conclude that w.p.1 we have

$$d_\infty(\Pi_{m_k} f_k^*, K) > (1-\delta_1)\psi_o(n_k) \quad \text{i.o.}$$

One may apply results from the theory of linear spaces. See, for example, Banach (1932) chapter XI § 4 or Köthe (1960) § 26.4. We indicate a simple proof using the structure of H and $\Pi_m H$.

It is well-known that H is isometric isomorphic with ℓ_2. Let $e = (e_0, \ldots, e_{m-1}) \in \mathbb{R}^m$ with $e_j = \varphi_j(1)$ and $\Pi_m f = \sum_{j=0}^{m-1} f_j \varphi_j$. Then $\Pi_m f(1) = \Sigma f_j e_j = (f, e)_m$, where $(\cdot,\cdot)_m$ is the inner product in $\mathbb{R}^m$. Lemma 2.1

implies that we may consider $(f,e)_m$ instead of $\|\Pi_m f\|_\infty$.

W.p.1 we have $(\Pi_{m_k} f_k^*, e)_{m_k} > 1 - \epsilon$ and $\|\Pi_{m_k} f_k^*\|_H > 1+(1-\delta)\psi_o(n_k)$. We write $\Pi_{m_k} f_k^* = ae + bd$ where $d \in \mathbb{R}^{m_k}$ perpendicular to e (i.e. $(e,d)_{m_k} = 0$). Thus $(\Pi_{m_k} f_k^*, e)_{m_k} = a > 1-\epsilon$. Then we have

$$d_\infty(\Pi_{m_k} f_k^*, K) \geq ((1 - \|\Pi_{m_k} f_k^*\|_H^{-1})\, \Pi_{m_k} f_k^*, e)_{m_k}$$

$$= (1-\| \Pi_{m_k} f_k^*\|_H^{-1})\; a\; (e,e)_{m_k} > (1-\delta_1)\psi_o(n_k).$$

To complete the proof we consider

$$P(\| \Pi_{m_k} f_k^*\|_H > 1+(1-\delta)\psi_o(n_k) \wedge \|f_k^* - \Pi_{m_k} f_k^*\|_\infty > (\psi_o(n_k)/m_k)^{1/2})$$

$$= P(\| \Pi_{m_k} f_k^*\|_H > 1+(1-\delta)\psi_o(n_k) P(\|f_k^* - \Pi_{m_k} f_k^*\|_\infty > (\psi_o(n_k)/m_k)^{1/2})$$

because of the independence of $\Pi_{m_k} f_k^*$ and $f_k^* - \Pi_{m_k} f_k^*$. Applying (3.1), lemma 2.2 and (2.3.b) we obtain

$$\sum_k P(\| \Pi_{m_k} f_k^*\|_H > 1+(1-\delta)\psi_o(n_k) \wedge \|f_k^* - \Pi_{m_k} f_k^*\|_\infty > (\psi_o(n_k)/m_k)^{1/2}) < \infty$$

Thus w.p.1 we have i.o.

$$\| \Pi_{m_k} f_k^*\|_H > 1+(1-\delta)\psi_o(n_k)$$

and

$$\| \Pi_{m_k} f_k^* - f_k^*\|_\infty \leq (\psi_o(n_k)/m_k)^{1/2}.$$

One easily sees that for k sufficiently large

$$(\psi_o(n_k)/m_k)^{1/2} < \frac{1}{3}\, \delta\psi_o(n_k).$$

4. Upper bound.

In this section we use the subsequences $n_k = \exp(k/\varphi(k))$ where $\varphi(k) = (\log k)^{4/3}$ and $m_k = (L_2 n_k)^{1/3}$. We write

$$P(d_\infty(f_{n_k},K) > \psi_\delta(n_k) =$$

$$P(d_\infty(f_{n_k},K) > \psi_\delta(n_k) \wedge \Pi_{m_k} f_k \notin K) +$$

$$P(d_\infty(f_{n_k},K) > \psi_\delta(n_k) \wedge \Pi_{m_k} f_{n_k} \in K \cap \overline{K}_{1-(L_2 n_k)^{-1/6}}) +$$

$$P(d_\infty(f_{n_k},K) > \psi_\delta(n_k) \wedge \Pi_{m_k} f_{n_k} \in K_{1-(L_2 n_k)^{-1/6}}) =$$

$$P_{1,k} + P_{2,k} + P_{3,k}.$$

The event in $P_{3,k}$ implies $f_{n_k} \in K$ for n_k sufficiently large. This follows from the following result.

Lemma 4.1. Let n_k and m_k be defined as above. Then

$$\sup_{m_k < m \leq m_{k+1}} \sup_{0 \leq t \leq 1} |U_m(t)| \, (2L_2 n_k)^{-1/2} \leq (L_2 n_k)^{-1/6} \quad \text{a.s.}$$

for k sufficiently large.

Proof. Using lemma 2.1.b we have

$$P(\sup_{0 \leq t \leq 1} U_{m_k}(t) > \tfrac{1}{2}\sqrt{2}\ (L_2 n_k)^{1/3}) \leq (1+o(1))P(U > \tfrac{1}{2}\pi(1-\epsilon)(L_2 n_k)^{1/2})$$

$$= O((\varphi(k)/k)^{\pi^2(1-\epsilon)^2/8}\,(Lk)^{-1/2}) \quad \text{by (2.5).}$$

The Borel-Cantelli lemma implies: w.p.1 for k sufficiently large

$$\sup_{0\le t\le 1} (2L_2n_k)^{-1/2} |U_{m_k}(t)| \le 1/2\,(L_2n_k)^{-1/6}.$$

For $m_k < m \le m_{k+1}$ we have $U_m(t) = \sum_{j=m}^{m_{k+1}-1} X_j\varphi_j(t) + U_{m_{k+1}}(t)$. Consider

$$P(\sup_{m_k<m\le m_k-1} \sup_{0\le t\le 1} (2L_2n_k)^{-1/2} |\sum_{j=m}^{m_{k+1}-1} X_j\varphi_j(t)| > 1/2\,(L_2n_k)^{-1/6})$$

$$\le \sum_{m=m_k}^{m_{k+1}} P(\sup_{0\le t\le 1} |\sum_{j=m}^{m_{k+1}-1} X_j\varphi_j(t)| > 1/2\,\sqrt{2}\,(L_2n_k)^{1/3})$$

$$\le c(m_{k+1}-m_k)P(|U| > 1/2\pi\,\sqrt{3}\,(1-\epsilon)k^{1/2}\,Lk).$$

Application of (2,5) and the Borel-Cantelli lemma gives the desired result. □

Lemma 4.2. Let n_k be defined as above. Then, w.p.1 and for k sufficiently large, we have

$$\max_{n_k<n\le n_{k+1}} \sup_{0\le t\le 1} |f_n(t) - f_{n_k}(t)| < \epsilon\psi_o(n_k).$$

Proof. From the definition of f_n we have

$$\text{(4.1)}\qquad \max_{n_k<n\le n_{k+1}} \sup_{0\le t\le 1} |f_n(t) - f_{n_k}(t)| =$$

$$\max_{n_k<n\le n_{k+1}} \sup_{0\le t\le 1} \left| \frac{W(nt)}{(2nL_2n)^{1/2}} - \frac{W(n_kt)}{(2n_kL_2n_k)^{1/2}} \right| \le$$

$$\max_{n_k<n\le n_{k+1}} \sup_{0\le t\le 1} \left| \frac{W(nt) - W(n_kt)}{(2nL_2n)^{1/2}} \right| +$$

$$\max_{n_k<n\le n_{k+1}} \left| \left[\frac{2n_kL_2n_k}{2nLn} \right)^{1/2} \right] - 1 \right| \sup_{0\le t\le 1} \frac{|W(n_kt)|}{(2n_kL_2n_k)^{1/2}}\ .$$

From the definition of n_k it follows that $(n_{k+1}-n_k)n_{k+1}^{-1} \sim \{\varphi(k)\}^{-1}$. Using the L.I.L. we have that the last term on the right hand side of (4.1) is less than $2\{\varphi(k)\}^{-1}$. Next we consider

$$P\Big(\max_{n_k<n\le n_{k+1}} \sup_{0\le t\le 1} (2nL_2n)^{-1/2} |W(nt)-W(n_kt)| > \epsilon\psi_o(n_k)\Big)$$

$$\le P\Big(\sup_{0\le u\le 1-h} \sup_{0\le u\le h} |W(u+v) - W(v)| > \epsilon_1\psi_o(n_k)(2L_2n_k)^{1/2}\Big)$$

using the scaling property of the Brownian motion where $h \sim (\log k)^{-4/3}$. Now we apply the estimate given in lemma 1.1.1 of Csörgö and Révész (1981) and the Borel-Cantelli lemma in order to obtain the desired result. □

The assertion in the last lemma gives us that we have only to show that f_{n_k} is close to K.

Lemma 4.3. Take $\delta > 0$. Let n_k and m_k be defined as above. Then

$$\| \Pi_{m_k} f_{n_k} \|_H < 1 + \psi_\delta(n_k) \quad \text{a.s.}$$

for k sufficiently large.

Proof. It follows from the definition of $\|\cdot\|_H$ that we have

$$P(\|\Pi_{m_k} f_{n_k}\|_H > 1 + \psi_\delta(n_k)) = P(\chi^2_{m_k} > (2L_2n_k)(1 + \psi_\delta(n_k))^2)$$

Applying lemma 2.2 we have that the summation of the probabilities converges. The Borel-Cantelli lemma gives the result. □

Now we can complete the proof for the upper bound. It follows from lemma 4.3 that in the events described in $P_{1,k}$ and $P_{2,k}$ we have, for $\delta > 0$,

$$1 - (L_2 n_k)^{-1/6} \le \| \Pi_{m_k} f_{n_k} \|_H \le 1 + (1+\delta)\psi_o(n_k) \quad \text{a.s.}$$

We have

$$P_{1,k} = P(d_\infty(f_{n_k},K) > (1+\delta)\psi_o(n_k) \wedge \Pi_{m_k} f_{n_k} \notin K)$$

$$\le P(d_\infty(f_{n_k},K) > (1+\delta)\psi_o(n_k) \wedge 1 \le \| \Pi_{m_k} f_{n_k} \|_H < 1 + (1 + \tfrac{1}{2}\delta)\psi_o(n_k)) + r_{n_k}$$

where the error r_{n_k} is given in the proof of lemma 4.3.

As we have seen in section 2 we can write

$$f_{n_k}(t) = \Pi_{m_k} f_{m_k}(t) + (2L_2 n_k)^{-1/2}\, U_{m_k}(t)$$

where $\Pi_{m_k} f_{n_k}(t)$ and $U_{m_k}(t)$ are independent. In lemma 4.1 we showed that $(2L_2 n_k)^{-1/2}\, U_{m_k}(t)$ is small.

$$P(d_\infty(f_{n_k},K) > (1+\delta)\psi_o(n_k) \wedge 1 \le \| \Pi_{m_k} f_{n_k} \|_H < 1 + (1+\tfrac{1}{2}\delta)\psi_o(n_k))$$

$$\le P(\| \Pi_{m_k} f_{n_k} \|_H \ge 1)\; P((2L_2 n_k)^{-1/2} |U_{m_k}(1)| > \tfrac{1}{2}\, \delta\psi_o(n_k))$$

$$= O(k^{-1}(Lk)^{7/6} e^{(L_2 k)^{1/3} L_3 k/3} \cdot e^{-\delta_1^2 \pi^2 (L_2 k)^2/8})$$

by lemma 2.2 and estimate (2.5). Since $\sum_k P_{1,k} < \infty$ the Borel Cantelli lemma gives the desired result.

For the event considered in $P_{2,k}$ we define the stopping time M by

$$\{M=m\} = \{\| \Pi_m f_{n_k} \|_H \le 1 + \psi_o(n_k) < \| \Pi_{m+1} f_{n_k} \|_H\} .$$

Then we have

$$P(M=m) = P(\chi_m^2 \le 2L_2 n_k\, (1+\psi_o(n_k))^2 \le \chi_m^2 + U^2)$$

where χ_m^2 and U^2 are independent and chi-square distributed with respect to m

and 1 degrees of freedom.

If $\Pi_{m_k} f_{n_k} \in K \cap \bar{K}_{1-(L_2 n_k)^{-1/6}}$ we have $M = m > m_k$. The Borel-Cantelli lemma gives us an upper bound for M.

$$P_k^* = P(1 - (L_2 n_k)^{-1/6} \le \| \Pi_{m_k} f_{n_k} \|_H \le \| \Pi_{m_k^{5/2}} f_{n_k} \|_H \le 1 + \psi_o(n_k))$$

$$= P(2L_2 n_k (1-(L_2 n_k)^{-1/6})^2 \le \chi^2_{m_k} \le \chi^2_{m_k^{5/2}} \le (2L_2 n_k)(1+\psi_o(n_k))^2)$$

$$\le P(\chi^2_{m_k^{5/2}} \le 2L_2 n_k (1-(L_2 n_k)^{-1/6})^2)\ P(B(\tfrac{1}{2} m_k, \tfrac{1}{2} m_k^{5/2} - \tfrac{1}{2} m_k) \ge (1-(L_2 n_k)^{-1/6})^2)$$

by conditioning on $\chi^2_{m_k^{5/2}}$ and lemma 2.3 parts b and c

$$= O((Lk)^{5/6} k^{-1}\, e^{2(L_2 n_k)^{5/6}}\, e^{-(L_2 n_k)^{2/3}}\, e^{\frac{1}{2} m_k^{5/2} L(2eL_2 n_k / m_k^{5/2})})$$

$$\cdot\ O(m_k^{-1/2}\, e^{3m_k L m_k/4}\, m_k^{-5/2}\, e^{-\frac{1}{2}(m_k^{5/2} - m_k) L((L_2 n_k)^{1/6}/2)})$$

by lemma 2.2 and computation of the probabilities of a beta distributed r.v. It follows from the upperbound as derived above that

$$\Sigma P_k^* < \infty.$$

The Borel-Cantelli lemma gives us that from now on we only have to consider those values for M that are less than $m_k^{5/2}$.

When $M = m$ we have $\Pi_m f_{n_k} \notin K$. Finally we shall prove that in this case we have

$$d_\infty(\Pi_m f_{n_k}, K) < \epsilon\psi_o(n_k)$$

and also

$$d_\infty(\Pi_m f_{n_k}, f_{n_k}) < \epsilon\psi_o(n_k).$$

Then we have proved the assertion for the upper bound. Consider

$$P_k^{**} = \sum_{m=m_k}^{m_k^{5/2}} P(\Pi_{m_k} f_{n_k} \in K \cap \bar{K}_{1-(L_2 n_k)^{-1/6}} \wedge M = m \wedge$$

$$|X_{m+1}\varphi_{m+1}(t)| > \epsilon\psi_o(n_k))$$

$$\leq \sum_{m=m_k}^{m_k^{5/2}} P(\chi^2_{m+1} \geq 2L_2 n_k(1+\psi_o(n_k))^2) \cdot P(B(\tfrac{1}{2},\tfrac{1}{2}m) \geq \epsilon_1 m\psi_o(n_k))$$

by conditioning on χ^2_{m+1}, lemma 2.3 parts b and c and we also use that $\chi^2_{m+1} \in [2L_2 n_k(1+\psi_o(n_k))^2, 2(1+\epsilon)L_2 n_k]$. Using the estimates for χ^2 and beta distribution we obtain $\sum_k P_k^{**} < \infty$.

Finally we consider

$$P_k^{***} = \sum_{m=m_k}^{m_k^{5/2}} P(\Pi_{m_k} f_{n_k} \in K \cap \bar{K}_{1-(L_2 n_k)^{-1/6}} \wedge M=m \wedge$$

$$\| U_{m+1}(\cdot)\|_\infty (2L_2 n_k)^{-1/2} \geq \epsilon\psi_o(n_k))$$

$$\leq \sum_{m=m_k}^{m_k^{5/2}} P(\chi^2_{m+1} \geq 2L_2 n_k(1+\psi_o(n_k))^2) P(\sup_{0\leq t\leq 1} |U_{m+1}(t)| > (2L_2 n_k)^{1/2}\epsilon\psi_o(n_k))$$

by the independence of the processes U_{m+1} and V_{m+1}. Using the lemmas 2.1 and 2.2 we obtain

$$\sum_k P_k^{***} < \infty.$$

5. <u>Discussion and remarks</u>.

All proofs of Strassen's theorem contain the following assertions.

i) There exists some sequence $\{n_k\}$ such that w.p.1 $d_\infty(f_n, f_{n_k}) < \epsilon$ for all

$n \in (n_k, n_{k+1})$ and k sufficiently large.

ii) Let $\Pi_m f$ be the piecewise linear approximation of f. Then, for fixed m, w.p.1 $d_\infty(\Pi_m f_{n_k}, f_{n_k}) < \epsilon$ for m,k sufficiently large.

iii) $\int_0^1 \{\frac{d}{dt} \Pi_m f_{n_k}(t)\}^2 dt$ has the same distribution as $(2L_2 n_k)^{-1} \chi_m^2$ and is w.p.1 less than $(1+\epsilon)^2$ for k sufficiently large. Thus the last assertion implies $\| \Pi_m f_{n_k} \|_H \leq 1 + \epsilon$ w.p.1 for k large.

The projection Π_m as defined in section 2 is a different approximation than the one above. Above we have that $\Pi_m f$ and $f-\Pi_m f$ are dependent. In the approximation used in sections 3 and 4 of this paper we have that $\Pi_m f_n = (2L_2 n)^{-1/2} V_m$ and $f_n - \Pi_m f_n = (2L_2 n)^{-1/2} U_m$ are independent.

As far as I know is m fixed in all proofs of the F.L.I.L.

<u>Remark 1</u>. In order to prove an integral test for the rate of convergence in the F.L.I.L. one needs asymptotic *expansions* in lemma 2.1. The lower class result becomes more difficult to prove.

<u>Remark 2</u>. For the Brownian bridge we have the following expansion

$$B(t) = \sum_{k=1}^{\infty} (k\pi)^{-1} X_k \sqrt{2} \sin(k\pi t) \qquad 0 \leq t \leq 1$$

where $X_1, X_2, \ldots$ are i.i.d. N(0,1). See Shorack and Wellner (1986) chapter 1 exercise 15. By the same method as given in this paper one can obtain the rate of convergence in Finkelstein's F.L.I.L. See Finkelstein (1971) or Shorack and Wellner (1986) chapter 13 section 3 theorem 1.

References

Banach, S.: Théorie des opérations lineaires. Z. Subwencji. Funduszu Kultury Narodowej, Warszawa (1932)

Berman, S.M.: The maximum of a Gaussian process with non-constant variance. Ann. Inst. Henri Poincaré. 21, 383-391 (1985)

Bolthausen, E.: On the speed of convergence in Strassen's law of the iterated logarithm. Ann. Probab. 6, 668-672 (1978)

Chover, J.: On Strassen's version of the log log law. Z. Wahrscheinlichkeitstheorie und Verw. Geb. 8, 83-90 (1967)

Csáki, E.: A Relation Between Chung's and Strassen's Laws of the Iterated Logarithm. Z. Wahrscheinlichkeitstheorie und Verw. Geb. 54, 287-301 (1980)

Csörgö, M. and P. Révész: Strong approximations in probability and statistics. Academic Press, New York (1981)

Feller, W.: An introduction to probability theory and its applications. Vol. II. Second edition. Wiley & Sons, New York (1971)

Finkelstein, H.: The law of the iterated logarithm for empirical distributions. Ann. Math. Statist. 42, 607-615 (1971)

Freedman, D.: Brownian motion and diffusion. Holden-Day, San Francisco (1971)

Grill, K.: On the Rate of Convergence in Strassen's Law of the Iterated Logarithm. Prob. Th. Rel. Fields 74, 583-589 (1987)

Jain, N.C. and M.B. Marcus: Continuity of sub Gaussian processes. Advances in Probability and related topics. Vol. 4. Probability in Banach spaces, Ed. J. Kuelbs (1978)

Kö he, G.: Topologische lineaire Räume I. Springer-Verlag, Berlin (1960)

Kuelbs, J. and R. LePage: The law of the iterated logarithm for Brownian motion in a Banach space. Trans. Amer. Math. Soc. 185, 253-264 (1973)

Loève, M.: Probability Theory 3d ed., Van Nostrand, Princeton, N.J. (1963)

Piterbarg, V.I. and V.P. Prisjaznjuk.: Asymptotics of the probability of large excursions for a non-stationary Gaussian process. Theor. Prob. Math. Statist. 19, 131-144 (1978)

Rohatgi, V.K.: An Introduction to Probability Theory and Mathematical Statistics. Wiley & Sons, New York (1976)

Shorack, G.R. and J.A. Wellner.: Empricial processes with applications to statistics. Wiley & Sons, New York (1986)

Strassen, V.: An invariance principle for the law of the iterated logarithm. Z. Wahrscheinlichkeitstheor. Ver. Geb. 3, 211-216 (1964)

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